ALGEBRA II

6.1 mathematical functions: In mathematics, a **function**[1] is a relation between a set of inputs and a set of permissible outputs with the property that each input is related to exactly one output. An example is the function that relates each real number x to its square x^2. The output of a function f corresponding to an input x is denoted by $f(x)$ (read "f of x"). In this example, if the input is −3, then the output is 9, and we may write $f(-3) = 9$. The input variable(s) are sometimes referred to as the argument(s) of the function.

Functions of various kinds are "the central objects of investigation"[2] in most fields of modern mathematics. There are many ways to describe or represent a function. Some functions may be defined by a formula or algorithm that tells how to compute the output for a given input. Others are given by a picture, called the graph of the function. In science, functions are sometimes defined by a table that gives the outputs for selected inputs. A function could be described implicitly, for example as the inverse to another function or as a solution of a differential equation.

The input and output of a function can be expressed as an ordered pair, ordered so that the first element is the input (or tuple of inputs, if the function takes more than one input), and the second is the output. In the example above, $f(x) = x^2$, we have the ordered pair (−3, 9). If both input and output are real numbers, this ordered pair can be viewed as the Cartesian coordinates of a point on the graph of the function. But no picture can exactly define every point in an infinite set.

In modern mathematics,[3] a function is defined by its set of inputs, called the *domain*; a set containing the set of outputs, and possibly additional elements, as members, called its *codomain*; and the set of all input-output pairs, called its *graph*. (Sometimes the codomain is called the function's "range", but **warning**: the word "range" is sometimes used to mean, instead, specifically the set of outputs. An unambiguous word for the latter meaning is the function's "image". To avoid ambiguity, the words "codomain" and "image" are the preferred language for their concepts.) For example, we could define a function using the rule $f(x) = x^2$ by saying that the domain and codomain are the real numbers, and that the graph consists of all pairs of real numbers (x, x^2). Collections of functions with the same domain and the same codomain are called function spaces, the properties of which are studied in such mathematical disciplines as real analysis, complex analysis, and functional analysis.

In analogy with arithmetic, it is possible to define addition, subtraction, multiplication, and division of functions, in those cases where the output is a number. Another important operation defined on functions is function composition, where the output from one function becomes the input to another function.

Introduction and examples

A function that associates to any of the four colored shapes its color. For an example of a function, let X be the set consisting of four shapes: a red triangle, a yellow rectangle, a green hexagon, and a red square; and let Y be the set consisting of five colors: red, blue, green, pink, and yellow. Linking each shape to its color is a function from X to Y: each shape is linked to a color (i.e., an element in Y), and each shape is "linked", or "mapped", to exactly one color. There is no shape that lacks a color and no shape that has two or more colors. This function will be referred to as the "color-of-the-shape function".

The input to a function is called the argument and the output is called the value. The set of all permitted inputs to a given function is called the domain of the function, while the set of permissible outputs is called the codomain. Thus, the domain of the "color-of-the-shape function" is the set of the four shapes, and the codomain consists of the five colors. The concept of a function does not require that every possible output is the value of some argument, e.g. the color blue is not the color of any of the four shapes in X.

A second example of a function is the following: the domain is chosen to be the set of natural numbers (1, 2, 3, 4, ...), and the codomain is the set of integers (..., −3, −2, −1, 0, 1, 2, 3, ...). The function associates to any natural number n the number 4−n. For example, to 1 it associates 3 and to 10 it associates −6. A third example of a function has the set of polygons as domain and the set of natural numbers as codomain. The function associates a polygon with its number of vertices. For example, a triangle is associated with the number 3, a square with the number 4, and so on.The term range is sometimes used either for the codomain or for the set of all the actual values a function has. To avoid ambiguity this article avoids using the term.

Definition

In order to avoid the use of the informally defined concepts of "rules" and "associates", the above intuitive explanation of functions is completed with a formal definition. This definition relies on the notion of the Cartesian product. The Cartesian product of two sets X and Y is the set of all ordered pairs, written (x, y), where x is an element of X and y is an element of Y. The x and the y are called the components of the ordered pair. The Cartesian product of X and Y is denoted by X × Y.

A function f from X to Y is a subset of the Cartesian product X × Y subject to the following condition: every element of X is the first component of one and only one ordered pair in the subset.[4] In other words, for every x in X there is exactly one element y such that the ordered pair (x, y) is contained in the subset defining the function f. This formal definition is a precise rendition of the idea that to each x is associated an element y of Y, namely the uniquely specified element y with the property just mentioned.

Considering the "color-of-the-shape" function above, the set X is the domain consisting of the four shapes, while Y is the codomain consisting of five colors. There are twenty possible ordered pairs (four shapes times five colors), one of which is ("yellow rectangle", "red").

The "color-of-the-shape" function described above consists of the set of those ordered pairs, (shape, color) where the color is the actual color of the given shape. Thus, the pair ("red triangle", "red") is in the function, but the pair ("yellow rectangle", "red") is not.

Specifying a function

A function can be defined by any mathematical condition relating each argument (input value) to the corresponding output value. If the domain is finite, a function f may be defined by simply tabulating all the arguments x and their corresponding function values f(x). More commonly, a function is defined by a formula, or (more generally) an algorithm — a recipe that tells how to compute the value of f(x) given any x in the domain.

There are many other ways of defining functions. Examples include piecewise definitions, induction or recursion, algebraic or analytic closure, limits, analytic continuation, infinite series, and as solutions to integral and differential equations. The lambda calculus provides a powerful and flexible syntax for defining and combining functions of several variables. In advanced mathematics, some functions exist because of an axiom, such as the Axiom of Choice.

Graph The graph of a function is its set of ordered pairs F. This is an abstraction of the idea of a graph as a picture showing the function plotted on a pair of coordinate axes; for example, (3, 9), the point above 3 on the horizontal axis and to the right of 9 on the vertical axis, lies on the graph of y=x2.

Formulas and algorithms

Different formulas or algorithms may describe the same function. For instance f(x) = (x + 1) (x − 1) is exactly the same function as f(x) = x2 − 1.[6] Furthermore, a function need not be described by a formula, expression, or algorithm, nor need it deal with numbers at all: the domain and codomain of a function may be arbitrary sets. One example of a function that acts on non-numeric inputs takes English words as inputs and returns the first letter of the input word as output.

As an example, the factorial function is defined on the nonnegative integers and produces a nonnegative integer. It is defined by the following inductive algorithm: 0! is defined to be 1, and n! is defined to be for all positive integers n. The factorial function is denoted with the exclamation mark (serving as the symbol of the function) after the variable (postfix notation).

Computability

Functions that send integers to integers, or finite strings to finite strings, can sometimes be defined by an algorithm, which gives a precise description of a set of steps for computing the output of the function from its input. Functions definable by an algorithm are called computable functions. For example, the Euclidean algorithm gives a precise process to compute the greatest common divisor of two positive integers. Many of the functions studied in the context of number theory are computable.

Fundamental results of computability theory show that there are functions that can be precisely defined but are not computable. Moreover, in the sense of cardinality, almost all functions from the integers to integers are not computable. The number of computable functions from integers to integers is countable, because the number of possible algorithms is. The number of all functions from integers to integers is higher: the same as the cardinality of the real numbers. Thus most functions from integers to integers are not computable. Specific examples of uncomputable functions are known, including the busy beaver function and functions related to the halting problem and other undecidable problems.

Variants and generalizations

Alternative definition of a function

The above definition of "a function from X to Y" is generally agreed on,[citation needed] however there are two different ways a "function" is normally defined where the domain X and codomain Y are not explicitly or implicitly specified. Usually this is not a problem as the domain and codomain normally will be known. With one definition saying the function defined by $f(x) = x2$ on the reals does not completely specify a function as the codomain is not specified, and in the other it is a valid definition.

In the other definition a function is defined as a set of ordered pairs where each first element only occurs once. The domain is the set of all the first elements of a pair and there is no explicit codomain separate from the image. Concepts like surjective have to be refined for such functions, more specifically by saying that a (given) function is surjective on a (given) set if its image equals that set. For example, we might say a function f is surjective on the set of real numbers.

If a function is defined as a set of ordered pairs with no specific codomain, then $f: X \to Y$ indicates that f is a function whose domain is X and whose image is a subset of Y. This is the case in the ISO standard.[7] Y may be referred to as the codomain but then any set including the image of f is a valid codomain of f. This is also referred to by saying that "f maps X into Y"[7] In some usages X and Y may subset the ordered pairs, e.g. the function f on the real numbers such that $y=x2$ when used as in $f: [0,4] \to [0,4]$ means the function defined only on the interval [0,2].[10] With the definition of a function as an ordered triple this would always be considered a partial function.

An alternative definition of the composite function $g(f(x))$ defines it for the set of all x in the domain of f such that $f(x)$ is in the domain of g.[11] Thus the real square root of $-x2$ is a function only defined at 0 where it has the value 0.

Functions are commonly defined as a type of relation. A relation from X to Y is a set of ordered pairs (x, y) with and . A function from X to Y can be described as a relation from X to Y that is left-total and right-unique. However when X and Y are not specified there is a disagreement about the definition of a relation that parallels that for functions. Normally a relation is just defined as a set of ordered pairs and a correspondence is defined as a triple (X, Y, F), however the distinction between the two is often blurred or a relation is never referred to without specifying the two sets. The definition of a function as a triple defines a function as a type of correspondence, whereas the definition of a function as a set of ordered pairs defines a function as a type of relation.

Many operations in set theory, such as the power set, have the class of all sets as their domain, and therefore, although they are informally described as functions, they do not fit the set-theoretical definition outlined above, because a class is not necessarily a set. However some definitions of relations and functions define them as classes of pairs rather than sets of pairs and therefore do include the power set as a function.

Partial and multi-valued functions is not a function in the proper sense, but a multi-valued function: it assigns to each positive real number x two values: the (positive) square root of x, and

In some parts of mathematics, including recursion theory and functional analysis, it is convenient to study partial functions in which some values of the domain have no association in the graph; i.e., single-valued relations. For example, the function f such that $f(x) = 1/x$ does not define a value for $x = 0$, since division by zero is not defined. Hence f is only a partial function from the real line to the real line. The term total function can be used to stress the fact that every element of the domain does appear as the first element of an ordered pair in the graph. In other parts of mathematics, non-single-valued relations are similarly conflated with functions: these are called multivalued functions, with the corresponding term single-valued function for ordinary functions.

Functions with multiple inputs and outputs

The concept of function can be extended to an object that takes a combination of two (or more) argument values to a single result. This intuitive concept is formalized by a function whose domain is the Cartesian product of two or more sets.

For example, consider the function that associates two integers to their product: $f(x, y) = x\cdot y$. This function can be defined formally as having domain $Z\times Z$, the set of all integer pairs; codomain Z; and, for graph, the set of all pairs $((x,y), x\cdot y)$. Note that the first component of any such pair is itself a pair (of integers), while the second component is a single integer.

The function value of the pair (x,y) is $f((x,y))$. However, it is customary to drop one set of parentheses and consider $f(x,y)$ a function of two variables, x and y. Functions of two variables may be plotted on the three-dimensional Cartesian as ordered triples of the form $(x,y,f(x,y))$.

The concept can still further be extended by considering a function that also produces output that is expressed as several variables. For example, consider the integer divide function, with domain $Z \times N$ and codomain $Z \times N$. The resultant (quotient, remainder) pair is a single value in the codomain seen as a Cartesian product.

Binary operations

The familiar binary operations of arithmetic, addition and multiplication, can be viewed as functions from $R \times R$ to R. This view is generalized in abstract algebra, where n-ary functions are used to model the operations of arbitrary algebraic structures. For example, an abstract group is defined as a set X and a function f from $X \times X$ to X that satisfies certain properties.

Traditionally, addition and multiplication are written in the infix notation: $x+y$ and $x \times y$ instead of $+(x, y)$ and $\times(x, y)$.

Functors

The idea of structure-preserving functions, or homomorphisms, led to the abstract notion of morphism, the key concept of category theory. In fact, functions $f: X \to Y$ are the morphisms in the category of sets, including the empty set: if the domain X is the empty set, then the subset of $X \times Y$ describing the function is necessarily empty, too. However, this is still a well-defined function. Such a function is called an empty function. In particular, the identity function of the empty set is defined, a requirement for sets to form a category.

The concept of categorification is an attempt to replace set-theoretic notions by category-theoretic ones. In particular, according to this idea, sets are replaced by categories, while functions between sets are replaced by functors.